Two-person non-zero-sum games as multicriteria goal games

F.R. Fernández[a], J. Puerto[a] and L. Monroy[b]

[a]*Departamento Estadística e Investigación Operativa. Facultad de Matemáticas, Universidad de Sevilla, E-41012 Sevilla, Spain*

[b]*Departamento Economía Aplicada I, Facultad de Económicas y Empresariales, Universidad de Sevilla, E-41012 Sevilla, Spain*

In this paper, we propose a new way to analyze bimatrix games. This new approach consists of considering the game as a bicriteria matrix game. The solution concepts behind this game are based on getting the probability to achieve some prespecified goals. We consider as a part of the solution, not only the payoff values, but also the probability to get them. In addition, to avoid the choice of only one goal, two different approaches are used. Firstly, sensitivity analysis of the solution set is carried out on the range of goals, secondly a partition of the goal space in a finite number of regions is presented. Some examples are included to illustrate the results in the paper.

Keywords: game theory, multicriteria games, goals, sensitivity analysis

1. Introduction

From the seminal paper of Nash [1], it is known that any bimatrix game has at least one equilibrium in mixed strategies. These equilibria for matrix games coincide with saddle points, introduced by von Neumann [2]. However, the property that all saddle points in matrix games have equal value is not verified by equilibria in bimatrix games. Non-zero-sum games may have two or more equally attractive equilibrium outcomes, each possessing equivalent status as a solution, with no compelling reason to choose among them. Thus, multiplicity of Nash equilibria poses a problem for the applicability of the concept of equilibrium as an unquestionable solution concept for these games. Several authors have suggested choosing, as a solution to the game, a subset of the equilibrium pairs with particular properties (see [4–8]).

As Luce and Raiffa [9] pointed out: “It is unfortunate that a unified theory for all non-cooperative games does not seem possible. The only alternative seems to be to complicate the problem by introducing more initial information in the form of boundary and initial conditions.”

Borm et al. [3] characterize the Pareto equilibria of bimatrix games. This concept induces a partial order in the set of equilibria based on vector domination. These equilibria need not be unique, therefore the initial problem still remains. Despite these difficulties, it is still possible to deal with this problem using an alternative analysis. As the payoffs in these games are vectors, it seems to be logical to analyze these situations from the beginning as a multicriteria decision problem.

The above comment could lead us to analyze bimatrix games as "bicriteria matrix games". To this end, Pareto-optimal security strategies (POSS) [10] have been proposed as a solution concept for these games, based on the similarity with security levels determined by saddle points in scalar matrix games. This concept is independent of the notion of equilibrium so that the opponent is only taken into account to establish the security levels for one's own payoffs. When it is used to select strategies, the concept of security levels has the important property that the payoff obtained by these strategies cannot be diminished by the opponent's deviation in strategy.

In [11], a methodology to get the whole set of POSS is developed, based on solving a multiple-criteria linear program. This approach shows the parallelism between these strategies in multicriteria games and minimax strategies in scalar zero-sum matrix games. This notion of security is based on expected payoffs. For this reason, only when the game is played many times can these strategies provide us with a real sense of security. On the other hand, if the game is played only once, as in one-shot games, a better analysis should consider not only the payoffs but also the probability to get them.

In this paper, we propose a new methodology to analyze bimatrix games. This methodology is based on Pareto-optimal security strategies whose security levels are defined for one of the players as the probability that self-determined goals may be achieved [12,13]. Thus, we study as a part of the solution concept not only the payoff values, but also the probability to get them. This means that the achieved payoff does not depend on the repetition of the game, but rather the risk levels the player is able to assume. For this reason, in this kind of analysis, goals represent a player's risk attitude: the higher the goals, the lower the probability of getting the payoffs.

Needless to say, the analysis depends on the risk attitude of the player who determines his/her goals. Nevertheless, if the player is not able to assume a concrete goal, it is still possible to analyze the game for all possible goal values. This kind of analysis is possible because it induces a finite partition in the solution space. Once the player has all of this information, a decision may be made and a strategy can be chosen. This procedure provides us with a general methodology to analyze bimatrix games which, in turn, leads to a concrete strategy as soon as the player is willing to assume his/her risk attitude.

The paper is organized as follows. In the following section, we present the model that we deal with and state the basic definitions that will be used in the rest of the paper. In section 3, we obtain the set of efficient strategies with respect to a goal game by means of a multiple linear problem. We cannot compare these strategies by the probability to get the different outcomes, but we may choose among them using some decision rules, such as choosing the strategy that maximizes the expected value.

In section 4, we analyze the sensitivity of the efficient solution set regarding modifications on the goals. Section 5 shows the decomposition of the goal space into regions. The efficient solution set induced by all the goals within each of these regions remains unchanged. This partition gives us the so-called “solution map”. With this procedure, the player has information about all possible outcomes of the game. Finally, section 6 presents our conclusions.

2. Model and definitions

Let $A = (a_{ij})$, $B = (b_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$, be the payoff matrices of a non-zero-sum game, and the mixed strategy spaces for player I (PI) and player II (PII), respectively:

$$X = \left\{ x \in \mathcal{R}^n, \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n \right\},$$

$$Y = \left\{ y \in \mathcal{R}^m, \sum_{j=1}^{m} y_j = 1, y_j \geq 0, j = 1, \ldots, m \right\}.$$

In this section, we analyze these games under player I’s point of view, using multicriteria goal games [12].

As player I’s decision affects the outcome of player II, this study can be made under two different attitudes. Regarding the first attitude, named “positive attitude”, player I tries to achieve the best outcome for both players. In the second attitude, named “negative attitude”, player I tries to obtain the best outcome for himself to the detriment of player II. Both approaches are similar; the only difference between them is that for the “positive attitude” we consider a bicriteria game with matrix (A, B), and for the “negative attitude” we consider a bicriteria game with matrix $(A, -B)$. Notice that this approach generalizes the method for obtaining optimal threat strategies for bimatrix games, introduced by Owen [14]. He gets these strategies by solving the zero-sum game whose payoff matrix is $A - B$, which in our approach is related to solving the bicriteria game with weight values equal to one. Let us analyze the “positive attitude”.

Let $P = (P_1, P_2)$ be goals specified by PI. These goals are the objectives that player I would like to achieve as the consequence of the game. They represent not only his/her desired payoff, but also his/her risk attitude. The goals should be fixed by the player. Nevertheless, if the player were not able to choose the goals at the beginning, there would exist a solution map which establishes a partition of the goal space into a finite number of regions where the solution set remains unchanged.

Definition 1. The expected payoff of the game with goals $P = (P_1, P_2)$, for each strategy pair $x \in X$ and $y \in Y$, is

$$\upsilon(x, y) = (\upsilon_1(x, y), \upsilon_2(x, y)),$$

where

$$\upsilon_1(x, y) = x^t A_P y, \quad \upsilon_2(x, y) = x^t B_P y,$$

$$A_P = (\delta_{ij}^1), \quad B_P = (\delta_{ij}^2), \quad 1 \le i \le n, 1 \le j \le m,$$

$$\delta_{ij}^1 = \begin{cases} 1 & \text{if } a_{ij} \ge P_1, \\ 0 & \text{otherwise;} \end{cases} \qquad \delta_{ij}^2 = \begin{cases} 1 & \text{if } b_{ij} \ge P_2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we associate each strategy pair (x, y) with the probability to get the values P_1 and P_2.

As PI plays against PII, every strategy $x \in X$ defines goal security levels for PI.

Definition 2. The *P-goal security level vector* of PI, for each $x \in X$, is

$$\upsilon^P(x) = (\upsilon_1^P(x), \upsilon_2^P(x)),$$

where

$$\upsilon_1^P(x) = \min_{y \in Y} \upsilon_1^P(x, y) = \min_{y \in Y} x^t A_P y = \min_j \left(\sum_{i=1}^n x_i \delta_{ij}^1 \right),$$

$$\upsilon_2^P(x) = \min_{y \in Y} \upsilon_2^P(x, y) = \min_{y \in Y} x^t B_P y = \min_j \left(\sum_{i=1}^n x_i \delta_{ij}^2 \right).$$

It is easy to see that $\upsilon_s^P(x)$, $s = 1, 2$, is the minimum probability to achieve a payoff at least of P_s when PI chooses strategy x, independently of player II.

Now, we establish a new analysis for bimatrix games based on goal security levels. We propose to choose strategies which are non-dominated with respect to the defined security levels in the modified game with matrices A_P, B_P. We call these strategies P-goal security strategies.

Definition 3. A strategy $x^* \in X$ is a *P-goal security strategy* (PGSS) for PI if there is no $x \in X$ such that $\upsilon^P(x^*) \le \upsilon^P(x)$, $\upsilon^P(x^*) \ne \upsilon^P(x)$.

These strategies have the property that with no other strategy can one have a higher joint probability of getting the payoffs given by the prespecified goals. For this reason, this approach is very convenient because it helps the players to make a decision (to play) by showing the probability they have to get the payoffs that they would like to have. We propose a characterization of PGSS using Multicriteria Linear Programming. In this way, we can easily obtain all PGSS for player I.

3. Determination of P-goal security strategies

Let us consider the following multiple-objective linear problem that we call the P-goal game linear multicriteria problem $(GLMP)_P$:

$$
\begin{array}{lll}
(GLMP)_P & \text{maximize} & \upsilon_1, \upsilon_2 \\
 & \text{subject to} & x^t A_P \geq (\upsilon_1, \ldots, \upsilon_1), \\
 & & x^t B_P \geq (\upsilon_2, \ldots, \upsilon_2), \\
 & & \sum_{i=1}^{n} x_i = 1, \\
 & & x \geq 0.
\end{array}
$$

Theorem 4. A strategy $x^* \in X$ is a PGSS and $\upsilon^* = (\upsilon_1, \upsilon_2)$ is its P-goal security level vector iff (υ^*, x^*) is an efficient solution of the problem $(GLMP)_P$.

Proof. Let x^* be a PGSS, then there is no $x \in X$ such that $\upsilon^P(x^*) \leq \upsilon^P(x)$, $\upsilon^P(x^*) \neq \upsilon^P(x)$. According to definition 2, it is equivalent to

$$\left(\min_j \left(\sum_{i=1}^{n} x_i \delta_{ij}^1\right), \min_j \left(\sum_{i=1}^{n} x_i \delta_{ij}^2\right)\right) \geq \left(\min_j \left(\sum_{i=1}^{n} x_i^* \delta_{ij}^1\right), \min_j \left(\sum_{i=1}^{n} x_i^* \delta_{ij}^2\right)\right),$$

$$\left(\min_j \left(\sum_{i=1}^{n} x_i \delta_{ij}^1\right), \min_j \left(\sum_{i=1}^{n} x_i \delta_{ij}^2\right)\right) \neq \left(\min_j \left(\sum_{i=1}^{n} x_i^* \delta_{ij}^1\right), \min_j \left(\sum_{i=1}^{n} x_i^* \delta_{ij}^2\right)\right),$$

Hence, x is an efficient solution of the problem

$$\max_{x \in X} \left(\min_j \left(\sum_{i=1}^{n} x_i \delta_{ij}^1\right), \min_j \left(\sum_{i=1}^{n} x_i \delta_{ij}^2\right)\right)$$

and this problem is equivalent to

$$
\begin{array}{lll}
(GLMP)_P & \text{maximize} & \upsilon_1, \upsilon_2 \\
 & \text{subject to} & x^t A_P \geq (\upsilon_1, \ldots, \upsilon_1), \\
 & & x^t B_P \geq (\upsilon_2, \ldots, \upsilon_2), \\
 & & \sum_{i=1}^{n} x_i = 1, \\
 & & x \geq 0.
\end{array}
$$

Conversely, assume that an efficient solution (υ^*, x^*) of $(GLMP)_P$ is not a PGSS. Then there exists $\bar{x} \in X$ such that

$$\left(\min_j\left(\sum_{i=1}^n \bar{x}_i\delta_{ij}^1\right), \min_j\left(\sum_{i=1}^n \bar{x}_i\delta_{ij}^2\right)\right) \geq \left(\min_j\left(\sum_{i=1}^n x_i^*\delta_{ij}^1\right), \min_j\left(\sum_{i=1}^n x_i^*\delta_{ij}^2\right)\right),$$

$$\left(\min_j\left(\sum_{i=1}^n \bar{x}_i\delta_{ij}^1\right), \min_j\left(\sum_{i=1}^n \bar{x}_i\delta_{ij}^2\right)\right) \neq \left(\min_j\left(\sum_{i=1}^n x_i^*\delta_{ij}^1\right), \min_j\left(\sum_{i=1}^n x_i^*\delta_{ij}^2\right)\right).$$

Taking $\bar{\upsilon} = (\bar{\upsilon}_1, \bar{\upsilon}_2)$, where $\bar{\upsilon}_1 = \min_j(\sum_{i=1}^n \bar{x}_i\delta_{ij}^1)$, $\bar{\upsilon}_2 = \min_j(\sum_{i=1}^n \bar{x}_i\delta_{ij}^2)$, the vector $(\bar{\upsilon}, \bar{x})$ is a feasible solution of $(GLMP)_P$ dominating (υ^*, x^*), and this is a contradiction because (υ^*, x^*) is an efficient solution of $(GLMP)_P$. □

This result establishes that the probabilities of achieving PI's prespecified goals are given by the efficient solutions set of problem $(GLMP)_P$.

Example 1. Let the payoff matrices of a non-zero sum game be

$$A = \begin{pmatrix} 6 & 3 & 2 & 8 \\ 4 & 9 & 7 & 2 \\ 8 & 2 & 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 7 & 8 & 1 \\ 9 & 2 & 4 & 4 \\ 4 & 8 & 3 & 5 \end{pmatrix}.$$

Suppose that player I has established goals $P = (6, 5)$. Under positive attitude, the P-goal security strategies and its P-goal security level vectors for PI are the convex hull of the solutions

$$(\upsilon^1, x^1) = (\upsilon_1^1, \upsilon_2^1; x_2^1, x_2^1, x_3^1) = (1/2, 1/4; 1/4, 1/2, 1/4),$$
$$(\upsilon^2, x^2) = (\upsilon_1^2, \upsilon_2^2; x_2^2, x_2^2, x_3^2) = (1/3, 1/3; 1/3, 1/3, 1/3).$$

If PI plays strategy $x^1 = (1/4, 1/2, 1/4)$, this means that he/she gets $P_1 = 6$ with a probability of at least 1/2, and PII get $P_2 = 5$ with a probability of at least 1/4.

The characterization given by theorem 4 allows us to develop several ways of scalarization for the multicriteria game, in order to choose among the whole set of PGSS. We consider the scalarization given through a scalar linear problem $P(\lambda)$ associated with $(GLMP)_P$:

$$
\begin{array}{lll}
P(\lambda) & \text{maximize} & \lambda_1\upsilon_1 + \lambda_2\upsilon_2 \\
& \text{subject to} & x^t A_P \geq (\upsilon_1, \ldots, \upsilon_1), \\
& & x^t B_P \geq (\upsilon_2, \ldots, \upsilon_2), \\
& & \sum_{i=1}^n x_i = 1, \\
& & x \geq 0,
\end{array}
$$

where $\lambda \in \Lambda^0 = \{\lambda \in \mathcal{R}^2 / \lambda > 0, \sum_{s=1}^2 \lambda_s = 1\}$.

Theorem 5. A strategy $x^* \in X$ is a PGSS and $\upsilon^* = (\upsilon_1^*, \upsilon_2^*)$ is its P-goal security level vector iff $\lambda^* \in \Lambda^0$ exists such that (υ^*, x^*) is an optimal solution of the problem $P(\lambda)$.

Proof. The proof is derived from the characterization of PGSS given in theorem 4 and the equivalence between efficient solutions of a multiobjective linear problem and the solutions of the associated weighted sum problems. □

Each component λ_s of the parameter $\lambda = (\lambda_1, \lambda_2) \in \Lambda^0$ can be interpreted as the relative importance that PI assigns to the corresponding scalar game with matrices A_P, B_P. Thus, if PI sets up fixed values for λ_s, the objective function of $P(\lambda)$ is perfectly determined. If PI chooses $\lambda_s = P_s$, $s = 1, 2$, this function is the expected value of goals $P = (P_1, P_2)$. This is a logical way to choose among PGSS, so PI may select a P-goal security strategy x^* that gives the highest expected value, i.e., the optimal solution of the scalar linear problem

$$
\begin{aligned}
\text{maximize} \quad & P_1\upsilon_1 + P_2\upsilon_2 \\
\text{subject to} \quad & x^t A_P \geq (\upsilon_1, \dots, \upsilon_1), \\
& x^t B_P \geq (\upsilon_2, \dots, \upsilon_2), \\
& \sum_{i=1}^{n} x_i = 1, \\
& x \geq 0.
\end{aligned}
$$

Example 2. Consider the payoff matrices and goals $P = (6, 5)$ of example 1. The P-goal security strategy that gives the maximum expected valued for goals P is $x^1 = (1/4, 1/2, 1/4)$.

4. Sensitivity analysis in the goals

Previously, we have obtained a P-goal security strategy and its P-goal security level vector solving a multiobjective linear problem. However, it may occur that the player is unable to determine “a priori” the risk level he/she is willing to assume, that is, to determine his/her goals. There is still a way to avoid this problem and, therefore, to apply this approach. As we state in the introduction, it is possible to establish a partition of the goal space into a finite number of sets. Each of these sets has the property that the final solutions are the same for any goal belonging to each of them. Thus, the player only has to compare a finite number of alternatives to decide with which strategy to play. The player has the advantage of knowing the probabilities assigned in order to achieve those goals.

We transform this problem into a sensitivity analysis of the solution sets of our *GLMP*. To this end, we want to determine whether an efficient solution for this problem remains efficient after changing the goals $P = (P_1, P_2)$. This analysis is of crucial importance because it gives the player a way of avoiding the problem of assuming an exact initial estimation of goals.

We consider two cases. In the first case, we assume that the goal $P = (P_1, P_2)$ increases to $P' = (P_1', P_2')$. In the second case, we assume that the goal P decreases to P'.

(1) If we increase P_1 to P_1' and P_2 to P_2', matrices $A_{P'}$ and $B_{P'}$ associated with goal $P' = (P_1', P_2')$ have more zero elements than matrices A_P and B_P, respectively. For this reason, the feasible set of the new linear problem $(GLMP)_{P'}$ is smaller. Hence, if (υ^*, x^*) is still feasible for the problem associated to goals P', it will be efficient for that problem.

We consider matrices $M_P^1 = (m_{ij}^1)$ and $M_P^2 = (m_{ij}^2)$, $1 \le i \le n$, $1 \le j \le m$, where

$$m_{ij}^1 = \begin{cases} 1 & \text{if } P_1 \le a_{ij} < P_1', \\ 0 & \text{otherwise;} \end{cases} \qquad m_{ij}^2 = \begin{cases} 1 & \text{if } P_2 \le b_{ij} < P_2', \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6. Let $(\upsilon_1^*, \upsilon_2^*, x^*)$ be an efficient solution of problem $(GLMP)_P$. If

$$\sum_{i=1}^{n} x_i^* m_{ij}^s \le h_j^{*s}, \quad j = 1, \ldots, m, \; s = 1, 2,$$

where h_j^{*1}, h_j^{*2} are the slack variables of the efficient solution, then $(\upsilon_1^*, \upsilon_2^*, x^*)$ is an efficient solution of problem $(GLMP)_{P'}$.

Proof. We can write $A_{P'} = A_P - M_P^1$, $B_{P'} = B_P - M_P^2$.

If $(\upsilon_1^*, \upsilon_2^*, x^*)$ is an efficient solution of problem $(GLMP)_P$, then

$$\begin{aligned} x^{*t} A_P &\ge (\upsilon_1^*, \ldots, \upsilon_1^*), \\ x^{*t} B_P &\ge (\upsilon_2^*, \ldots, \upsilon_2^*), \\ \sum_{i=1}^{n} x_i^* &= 1, \\ x^* &\ge 0. \end{aligned}$$

As we assume by hypothesis that

$$\sum_{i=1}^{n} x_i^* m_{ij}^s \le h_j^{*s}, \quad j = 1, \ldots, m, \; s = 1, 2,$$

these expressions can be rewritten as

$$\begin{aligned} x^{*t} M_P^1 &\le x^{*t} A_P - (\upsilon_1^*, \ldots, \upsilon_1^*), \\ x^{*t} M_P^2 &\le x^{*t} B_P - (\upsilon_2^*, \ldots, \upsilon_2^*), \end{aligned}$$

which means

$$(\upsilon_1^*, \quad , \upsilon_1^*) \leq x^{*t} A_P - x^{*t} M_P^1 = x^{*t}(A_P - M_P^1) = x^{*t} A_{P'},$$
$$(\upsilon_2^*, \quad , \upsilon_2^*) \leq x^{*t} B_P - x^{*t} M_P^2 = x^{*t}(B_P - M_P^2) = x^{*t} B_{P'}$$

and

$$\sum_{i=1}^{n} x_i^* = 1, \quad x^* \geq 0,$$

then $(\upsilon_1^*, \upsilon_2^*, x^*)$ is an efficient solution of problem $(GLMP)_{P'}$. □

(2) We now assume that goal P_1 decreases to P_1' and P_2 decreases to P_2'. In this case, matrices $A_{P'}$ and $B_{P'}$ have new elements with a value of 1, then the feasible set of the problem associated with goal P' is larger. For this reason, if (υ^*, x^*) is an efficient solution for the problem $(GLMP)_P$, it is still a feasible solution for the problem $(GLMP)_{P'}$, but may not be an efficient solution. To check whether (υ^*, x^*) is an efficient solution for the new problem, subproblem testing can be used. Let $A_{P'}$ and $B_{P'}$ be the matrices induced by P'. The new problem is

$$(GLMP)_{P'} \qquad \begin{array}{ll} \text{maximize} & \upsilon_1, \upsilon_2 \\ \text{subject to} & x^t A_{P'} \geq (\upsilon_1, \quad , \upsilon_1), \\ & x^t B_{P'} \geq (\upsilon_2, \quad , \upsilon_2), \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \geq 0. \end{array}$$

We can write $A_{P'} = A_P + M_P^1$ and $B_{P'} = B_P + M_P^2$, where $M_P^1 = (m_{ij}^1)$, $M_P^2 = (m_{ij}^2)$, $1 \leq i \leq n$, $1 \leq j \leq m$ are matrices whose elements are

$$m_{ij}^1 = \begin{cases} 1 & \text{if } P_1' \leq a_{ij} < P_1, \\ 0 & \text{otherwise;} \end{cases} \qquad m_{ij}^2 = \begin{cases} 1 & \text{if } P_2' \leq b_{ij} < P_2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, problem $(GLMP)_{P'}$ can be written as

$$(GLMP)_{P'} \qquad \begin{array}{ll} \text{maximize} & \upsilon_1, \upsilon_2 \\ \text{subject to} & x^t (A_P + M_P^1) \geq (\upsilon_1, \quad , \upsilon_1), \\ & x^t (B_P + M_P^2) \geq (\upsilon_2, \quad , \upsilon_2), \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \geq 0. \end{array}$$

Theorem 7. Let $(\upsilon_1^*, \upsilon_2^*, x^*)$ be an efficient solution of problem $(GLMP)_P$. If the scalar linear problem

$$\begin{aligned} \text{maximize} \quad & t_1 + t_2 \\ \text{subject to} \quad & x^t(A_P + M_P^1) \geq (\upsilon_1, \ldots, \upsilon_1), \\ & x^t(B_P + M_P^2) \geq (\upsilon_2, \ldots, \upsilon_2), \\ & \upsilon_s - t_s = \upsilon_s^*, \quad s = 1, 2, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \geq 0 \end{aligned}$$

has an optimal value of zero, then $(\upsilon_1^*, \upsilon_2^*, x^*)$ is an efficient solution for $(GLMP)_{P'}$.

Proof. If the optimal objective value of this problem is zero, then $t_1 = 0$ and $t_2 = 0$. That means, using the subproblem test for efficient points of Steuer [15], the solution $(\upsilon_1^*, \upsilon_2^*, x^*)$ cannot be improved component-wise. This fact implies that $(\upsilon_1^*, \upsilon_2^*, x^*)$ remains efficient in the new problem $(GLMP)_{P'}$. □

5. Partition of the goal space

Based on the sensitivity analysis developed in the previous section, we present in this section the partition of the goal space (GS) already commented on. We break GS into a finite number of rectangular regions such that for all goals in one of these regions, we obtain the same solution set. With this procedure, PI has information about all possible outcomes of the game. Therefore, the proposed analysis can always be applied even without the knowledge of “a priori” goals. This is because the player only has to compare a finite number of alternatives. Any of the well-developed methodologies of multicriteria decision making [16] may help him/her to make the decision.

Let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ be, respectively, the entries of matrices A and B ranked in increasing order. The different regions in the partition of the goal space are

$$\begin{aligned} R_{11} &= \{(\alpha_1, \beta_1)\}, \\ R_{1j} &= \{\alpha_1\} \times (B_{j-1}, \beta_j], \qquad j = 2, \ldots, s, \\ R_{i1} &= (\alpha_{i-1}, \alpha_i] \times \{\beta_1\}, \qquad i = 2, \ldots, r, \\ R_{ij} &= (\alpha_{i-1}, \alpha_i] \times (\beta_{j-1}, \beta_j], \qquad i = 2, \ldots, r, \; j = 2, \ldots, s. \end{aligned}$$

It should be noted that these regions correspond to rectangles, possibly degenerated to segments R_{1j}, R_{i1}, or to the point R_{11}.

If $P = (P_1, P_2) \in GS$, we denote by C_P the efficient solution set of problem $(GLMP)_P$:

$$C_P = \{(\upsilon_1^P, \upsilon_2^P, x^P) \text{ efficient solution of } (GLMP)_P\}.$$

Theorem 8. The following statements hold:

(1) For all P, P' belonging to R_{ij}, $C_P = C_{P'}$.

(2) Let $P = (P_1, P_2)$ be a goal that belongs to R_{ij} for a fixed i and $j = 1,\dots, m$. Let $\bar{\upsilon}_1^P$ be the value of the zero-sum matrix game with payoff matrix A_P, and let us denote by X^{P_1} the whole set of optimal strategies for this game. Then $(\bar{\upsilon}_1^P, \bar{\upsilon}_2^P, \bar{x}^P)$ is an efficient solution of $(GLMP)_P$, where

$$\bar{\upsilon}_2^P = \max_{x \in X^{P_1}} \min_j \left(\sum_{i=1}^n x_i^P \delta_{ij}^2 \right),$$

$$\bar{x}^P \in \arg \max_{x \in X^{P_1}} \min_j \left(\sum_{i=1}^n x_i^P \delta_{ij}^2 \right).$$

Proof. (1) By definition, for all $P, P' \in R_{ij}$, $A_P = A_{P'}$ and $B_P = B_{P'}$, then $(GLMP)_P = (GLMP)_{P'}$ and $C_P = C_{P'}$.

(2) For any $P_1 \in (\alpha_{i-1}, \alpha_i]$ and $P_2 \in (\beta_{j-1}, \beta_j]$, $j = 2,\dots, s$, let us consider the problem

$$\begin{aligned} &\text{maximize} && \upsilon_1, \upsilon_2 \\ &\text{subject to} && x^t A_P \geq (\upsilon_1, \dots, \upsilon_1), \\ &&& x^t B_P \geq (\upsilon_2, \dots, \upsilon_2), \\ &&& \sum_{i=1}^n x_i = 1, \\ &&& x \geq 0. \end{aligned}$$

If $(\bar{\upsilon}_1^P, \bar{x}^P)$ is an optimal solution of the scalar linear problem

$$\begin{aligned} &\text{maximize} && \upsilon_1 \\ &\text{subject to} && x^t A_P \geq (\upsilon_1, \dots, \upsilon_1), \\ &&& \sum_{i=1}^n x_i = 1, \\ &&& x \geq 0, \end{aligned}$$

then, taking $\bar{\upsilon}_2^P = \min_j (\sum_{i=1}^n x_i^P \delta_{ij}^2)$, we obtain that $(\bar{\upsilon}_1^P, \bar{\upsilon}_2^P, \bar{x}^P)$ is a lexicographical solution of $(GLMP)_P$. Hence, it is an efficient solution of problem $(GLMP)_P$. □

Remark 1. Given $P = (P_1, P_2) \in R_{ij}$, there is an efficient solution of problem $(GLMP)_P$ such that it gives the maximum probability of getting P_1, with $P_1 \in (\alpha_{i-1}, \alpha_i]$, for any P_2. In the same way, there is an efficient solution of problem $(GLMP)_P$ which gives the maximum probability to get P_2, with $P_2 \in (\beta_{j-1}, \beta_j]$, for any P_1. If problem $(GLMP)_P$ has only one efficient solution, then this solution gives the maximum joint probability of getting P_1 and P_2.

We have broken GS into rectangular regions R_{ij} such that all the goals in these regions can be obtained with the same probability. Nevertheless, it may happen that goals from different regions lead us to problems with the same solution sets.

Example 3. Let the payoff matrices of a non-zero-sum game be

$$A = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 0 & 2 \end{pmatrix}.$$

If we consider the positive attitude, the partition of the goal space is given by the following solutions map (see table 1).

Table 1

Solution map of example 3.

3			
(1, 0; 1, 0, 0) (1, 0; 0, 0, 1) (1, 0; 0, 1, 0) **2**	(1, 0; 1, 0, 0)	(1, 0; 1, 0, 0)	(1/2, 0; 1/2, 0, 1/2)
(1/2, 1/2; 1/2, 1/2, 0) (1, 1/2; 0, 1/2, 1/2)	(1/2, 1/2; 1/2, 1/2, 0) (1/2, 1/2; 0, 1/2, 1/2) (1,0;1,0,0)	(1/2, 1/2; 1/2, 1/2, 0) (1, 0; 1, 0, 0)	(1/4, 1/2; 1/4, 1/2, 1/4) (1/2, 0; 1/2, 0, 1/2)
0 (1, 1; 1, 0, 0) (1, 1; 0, 1, 0) (1, 1; 0, 0, 1)	**1** (1, 1; 1, 0, 0)	**2** (1, 1; 1, 0, 0)	**3** (1/2, 1; 1/2, 0, 1/2)

Inside each region, we have written the efficient extreme solutions which give the probabilities of getting goals belonging to each of them, and their corresponding strategies. As we can see, goals from different regions can be obtained with the same probabilities and the same strategies, therefore these regions may be collapsed (see table 2).

Table 2

Solution map of example 3.

3			
(1, 0; 1, 0, 0) (1 ,0; 0, 1, 0) (1, 0; 0, 0, 1) **2**	(1, 0; 1, 0, 0)		(1/2, 0; 1/2, 0, 1/2)
(1/2, 1/2; 1/2, 1/2, 0) (1, 1/2; 0, 1/2, 1/2)	(1/2, 1/2; 1/2, 1/2, 0) (1/2, 1/2; 0, 1/2, 1/2) (1,0;1,0,0)	(1/2, 1/2; 1/2, 1/2, 0) (1, 0; 1, 0, 0)	(1/4, 1/2; 1/4, 1/2, 1/4) (1/2, 0; 1/2, 0, 1/2)
0 (1, 1; 1, 0, 0) (1, 1; 0, 1, 0) (1, 1; 0, 0, 1)	**1** (1, 1; 1, 0, 0)	**2** (1, 1; 1, 0, 0)	**3** (1/2, 1; 1/2, 0, 1/2)

Based on theorem 8, we establish a procedure to obtain the set C_P, $\forall P$ GS. In order to do this, we denote by $(GLMP)_P(i, j)$ the problem $(GLMP)_{(\alpha_i, \beta_j)}$.

The procedure consists of the following steps:

1. Consider goal $P = (\alpha_1, \beta_1)$, and solve the problem $(GLMP)_P(1, 1)$. The extreme efficient solutions are the pure strategies of player I and the value of the game is given by $\upsilon_1 = 1$, $\upsilon_2 = 1$.
2. Consider goal $P = (\alpha_2, \beta_1)$ and problem $(GLMP)_P(2, 1)$. This problem has more constraints than $(GLMP)_P(1, 1)$; then we only need to check if the extreme efficient solutions of the last problem verify these constraints. Three cases can occur:
   2.1. If all extreme solutions verify these constraints, the solutions are still extreme efficient solutions of problem $(GLMP)_P(2, 1)$.
   2.2. If some of the extreme solutions verify the constraints but others do not, the new efficient solutions of problem $(GLMP)_P(2, 1)$ are in the boundary generated by the new constraints.
   2.3. If any efficient solution of problem $(GLMP)_P(1, 1)$ does not verify the new constraints of problem $(GLMP)_P(2, 1)$, all the efficient solutions of this problem are in the boundary generated by the new constraints.
3. Consider goal $P = (\alpha_3, \beta_1)$ and repeat step 2 with problems $(GLMP)_P(2, 1)$ and $(GLMP)_P(3, 1)$.

If we repeat this procedure in an orderly way for all goals, using the information obtained in each step (efficient basis), we can obtain an iterative method which gives us the efficient solutions of the new problem.

6. Conclusions

A new way to analyze two-person non-zero-sum games has been introduced. This analysis is based on considering these games as bicriteria zero-sum matrix games. Using goals, we consider as a solution not only the strategy played by the player, but also the probability of getting at least these goal values. Therefore, with this approach, each player has information about the probability of achieving the possible outcomes of the game. Thus, he/she does not take threats into account because the player knows the probability of achieving the different goals. In addition, the player knows the probability of getting different outcomes if cooperation exists. Therefore, this approach is useful to point out that players will do better if they cooperate.

References

[1] J. Nash, Noncooperative games, Annals of Mathematics 54(1951)289–285.
[2] J. von Neumann, Zur Theory der Gesellschaftsspiele, Mathematische Annalen 100(1928)295–320.
[3] P.E.M. Borm, M.J.M. Jansen, J.A.M. Potters and S.H. Tijs, Pareto equilibria for bimatrix games, Computers Math. Appl. 25(1993)19–25.
[4] J. Hillas, On the definition of the strategic stability of equilibria, Econometrica 58(1990)1365–1391.
[5] J.F. Mertens, Stable equilibria, a reformulation, part I: Definitions and basic properties, Mathematics of Operations Research 14(1989)575–625.
[6] J.F. Mertens, Stable equilibria, a reformulation, part II: Discussion of the definition and further results, Mathematics of Operations Research 16(1991)694–753.
[7] R.B. Myerson, Refinement of the Nash equilibrium concept, International Journal of Game Theory 7(1978)73–80.
[8] R. Selten, Reexamination of the perfectness concept for equilibrium points in extensive games, International Journal of Game Theory 4(1975)25–55.
[9] R.D. Luce and H. Raiffa, *Games and Decisions*, Wiley, New York, 1957.
[10] D. Ghose and U.R. Prasad, Solution concepts in two-person multicriteria games, Journal of Optimization Theory and Applications 63(1989)167–189.
[11] F.R. Fernández and J. Puerto, Vector linear programming in zero-sum multicriteria matrix games, Journal of Optimization Theory and Applications 89(1996)115–127.
[12] F.R. Fernández, L. Monroy and J. Puerto, Multicriteria goal games, presented at the *3rd Conference of Operations Research*, March 1997, La Habana, Cuba.
[13] L. Monroy, Análisis de juegos bipersonales con pagos vectoriales, Ph.D. Thesis, University of Sevilla, 1996.
[14] G. Owen, *Game Theory*, Academic Press, New York, 1982.
[15] R.E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Application*, Wiley, New York, 1986.
[16] V. Chankong and Y. Haimes, *Multiobjective Decision Making. Theory and Methodology*, North-Holland, New York, 1983.